Coefficient estimates for close-to-convex functions with argument β

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Abstract

This paper deals with coefficient estimates for close-to-convex functions with argument β ($-\pi/2 < \beta < \pi/2$). By using Herglotz representation formula, sharp bounds of coefficients are obtained. In particluar, we solve the problem posed by A. W. Goodman and E. B. Saff in [2]. Finally some complicted computations yield the explicit estimate of the third coefficient.

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1 Introduction

Let \mathcal{A} be the family of functions f analytic in the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, and \mathcal{A}_1 be the subset of \mathcal{A} consisting of functions f which are normalized by f(0)=f'(0)-1=0. A function $f\in\mathcal{A}_1$ is said to be starlike (denoted by $f\in\mathcal{S}^*$) if f maps \mathbb{D} univalently onto a domain starlike with respect to the origin. Let

$$\mathcal{P}_{\beta} = \left\{ p \in \mathcal{A} : p(0) = 1, \operatorname{Re} e^{i\beta} p > 0 \right\}.$$

Here and hereafter we always suppose $-\pi/2 < \beta < \pi/2$. It is easy to see that

$$p \in \mathcal{P}_{\beta} \Leftrightarrow \frac{e^{i\beta}p - i\sin\beta}{\cos\beta} \in \mathcal{P}_{0}.$$
 (1)

Herglotz representation formula (see [4]) together with (1) yield the following equivalence

$$p \in \mathcal{P}_{\beta} \Leftrightarrow p(z) = \int_{\partial \mathbb{D}} \frac{1 + e^{-2i\beta}xz}{1 - xz} d\mu(x)$$
 (2)

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for a Borel probability measure μ on the boundary $\partial \mathbb{D}$ of \mathbb{D} . This correspondence is 1-1.

Since \mathcal{P}_0 is the well-known Carathéodory class, we call \mathcal{P}_{β} the tilted Carathéodory class by angle β . Some equivalent definitions and basic estimates are known (for a short survey, see [7]).

Definition 1 A function $f \in A_1$ is said to be close-to-convex (denoted by $f \in CL$) if there exist a starlike function g and a real number $\beta \in (-\pi/2, \pi/2)$ such that

$$\frac{zf'}{g} \in \mathcal{P}_{\beta}.$$

This definition involving a real number β is slightly different from the original one due to Kaplan [5]. An equivalent definition of \mathcal{CL} by using Kaplan class and some related sets of univalent functions can be found in [6]. If we specify the real number β in the above definition, the corresponding function is called a close-to-convex function with argument β and we denote the class of all such functions by $\mathcal{CL}(\beta)$ (see [1, II, Definition 11.4]). Note that the union of class $\mathcal{CL}(\beta)$ over $\beta \in (-\pi/2, \pi/2)$ is precisely \mathcal{CL} while the intersection is the class of convex functions. These results were given in [2] without proof. Since the former one is obvious, we will only give an outline of the proof of the latter one. Choose a sequence $\{\beta_n\} \subset (-\pi/2, \pi/2)$ such that $\beta_n \to \pi/2$ as $n \to \infty$. The assertion follows from the facts that the class of starlike functions is compact in the sense of locally uniform convergence and any function sequence $\{p_n\}$ where $p_n \in \mathcal{P}_{\beta_n}$ converges to the constant function 1 locally uniform as $\beta_n \to \pi/2$.

In the literature, when studying the close-to-convex functions, some authors focus only on the case $\beta=0$. A. W. Goodman and E. B. Saff [2] were the first to point out explicitly that $\mathcal{CL}(\beta)$ and \mathcal{CL} are different when $\beta\neq 0$ and more deeply the class $\mathcal{CL}(\beta)$ has no inclusion relation with respect to β . Therefore it is useful to consider the individual class $\mathcal{CL}(\beta)$. The present paper follows their way in this direction and improves their result concerning the class $\mathcal{CL}(\beta)$;

Theorem A (Goodman-Saff [2]) Suppose
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CL}(\beta)$$
 for a $\beta \in (-\pi/2, \pi/2)$. Then

$$|a_n| \le 1 + (n-1)\cos\beta.$$

for $n = 2, 3, \dots$. If either n = 2 or $\beta = 0$, the inequality is sharp.

In the above mentioned paper, they also stated that the problem of finding the maximum for $|a_n|$ in the class $\mathcal{CL}(\beta)$ was difficult for $n \geq 3$. With regard to their problem, in the present paper we shall establish the following theorems:

Theorem 1 Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CL}(\beta)$ for $\alpha \beta \in (-\pi/2, \pi/2)$, then the sharp inequality

$$|a_n| \le \frac{2\cos\beta}{n} \max_{|u|=1} \left| \frac{n}{1 + e^{-2i\beta}} + \sum_{k=1}^{n-1} k u^{n-k} \right|.$$
 (3)

holds for $n = 2, 3, \dots$. Extremal functions are given by

$$f'(z) = \frac{1}{(1 - yz)^2} \frac{1 + e^{-2i\beta}yu_nz}{1 - yu_nz}$$

for $y \in \partial \mathbb{D}$, where $u_n \in \partial \mathbb{D}$ is a point at which the above maximum is attained.

We mention here that it seems that there are no extremal functions other than the form given above in Theorem 1. Theorem A follows from Theorem 1 immediately by the elementary inequality

$$\left| \frac{n}{1 + e^{-2i\beta}} + \sum_{k=1}^{n-1} k u^{n-k} \right| \le \frac{n}{2\cos\beta} + \frac{n(n-1)}{2}$$

for any $u \in \partial \mathbb{D}$.

The expression in (3) is implicit. When n=3, we can give a more concrete estimate and also show the extremal functions are unique;

Theorem 2 Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CL}(\beta)$, then the sharp inequality

$$|a_3| \le \frac{2\cos\beta}{3}\sqrt{5 + \frac{9}{4\cos^2\beta} + \frac{13}{1 - t_0}}\tag{4}$$

holds, where t_0 is the unique root of the equation

$$t^{3} - \left(\frac{4}{3}\cos^{2}\beta + 6\right)t^{2} + \left(\frac{40}{9}\cos^{2}\beta + 9\right)t + 4\cos^{2}\beta - 4 = 0$$
 (5)

in $0 \le t < 1$. Equality holds in (4) if and only if

$$f'(z) = \frac{1}{(1 - yz)^2} \frac{1 + e^{-2i\beta}yu_3z}{1 - yu_3z}$$

for some $y \in \partial \mathbb{D}$, where

$$u_3 = \begin{cases} 1 - \frac{t_0}{2} - i\sqrt{t_0 - \frac{t_0^2}{4}} \frac{\beta}{|\beta|}, & \text{when } \beta \neq 0; \\ 1, & \text{when } \beta = 0. \end{cases}$$

Remark 1 Comparing Theorem A and Theorem 2, it is not difficult to see that

$$1 + 2\cos\beta = \frac{2\cos\beta}{3}\sqrt{5 + \frac{9}{4\cos^2\beta} + \frac{13}{1 - t_0}}$$

if and only if

$$t_0 = \frac{9 - 9\cos\beta}{9 + 4\cos\beta}.$$

Since this t_0 is a root of (5) in [0,1) only when $\beta = 0$, Theorem A is sharp only when $\beta = 0$ for n = 3.

Finally we give an example to show how Theorem 2 works.

Example. Let $\beta = \pi/4$. Applying Mathematica, we may get the root of equation (5) which belongs to [0,1) is $0.201\cdots$, therefore in this case

$$|a_3| \lesssim 2.394$$

which is less than $1 + \sqrt{2} \approx 2.414$ by Theorem A.

2 Proof of Theorems

In order to prove our theorems, we shall need the following lemma

Lemma 1 (see [3] p. 52) If $f \in S^*$, then there exists a Borel probability measure ν on $\partial \mathbb{D}$ such that

$$f(z) = \int_{\partial \mathbb{D}} \frac{z}{(1 - yz)^2} d\nu(y).$$

Proof of Theorem 1:

Equivalence (2) and Lemma 1 imply that if $f \in \mathcal{CL}(\beta)$, then there exist two Borel probability measures μ and ν on $\partial \mathbb{D}$ such that f' can be represented as

$$f'(z) = \int_{\partial \mathbb{D}} \int_{\partial \mathbb{D}} \frac{1}{(1 - yz)^2} \frac{1 + e^{-2i\beta}xz}{1 - xz} d\mu(x) d\nu(y).$$

Thus in order to estimate the coefficients of f, it is sufficient to estimate those of functions

$$\frac{1}{(1-yz)^2} \frac{1 + e^{-2i\beta}xz}{1 - xz}$$

when |x| = |y| = 1.

Since

$$\frac{1}{(1-yz)^2} \frac{1+e^{-2i\beta}xz}{1-xz} = \sum_{n=0}^{\infty} \left\{ (n+1)y^n + \sum_{k=0}^{n-1} (k+1)(1+e^{-2i\beta})y^k x^{n-k} \right\} z^n$$

implies

$$|na_n| \le \max_{|x|=|y|=1} \left| ny^{n-1} + \sum_{k=0}^{n-2} (k+1)(1+e^{-2i\beta})y^k x^{n-1-k} \right|$$

$$= \max_{|x|=|y|=1} \left| n + \sum_{k=1}^{n-1} k(1+e^{-2i\beta})(x/y)^{n-k} \right|$$

after letting u = x/y, we can easily obtain (3). The extremal functions can be obtained easily by the proof of this theorem.

Proof of Theorem 2: By Theorem 1, we have the sharp inequality

$$|a_3| \le \frac{2\cos\beta}{3} \max_{-\pi < \alpha \le \pi} \sqrt{h(\alpha)}.$$

where

$$h(\alpha) = \left| 1 + 2e^{i\alpha} + \frac{3}{1 + e^{-2i\beta}} e^{2i\alpha} \right|^2.$$
 (6)

Straightforward calculations give

$$h(\alpha) = 5 + \frac{9}{4\cos^2\beta} + 4\cos\alpha + \frac{3\cos(\beta + 2\alpha) + 6\cos(\beta + \alpha)}{\cos\beta}$$

= $5 + \frac{9}{4\cos^2\beta} + (10\cos\alpha + 3\cos2\alpha) - 3\tan\beta(\sin2\alpha + 2\sin\alpha),$ (7)

and

$$h'(\alpha) = -4\sin\alpha - \frac{12\sin\frac{2\beta + 3\alpha}{2}\cos\frac{\alpha}{2}}{\cos\beta}$$

= $-(10\sin\alpha + 6\sin2\alpha) - 6\tan\beta(\cos2\alpha + \cos\alpha),$ (8)

$$h''(\alpha) = -(10\cos\alpha + 12\cos 2\alpha) + 6\tan\beta(2\sin 2\alpha + \sin\alpha). \tag{9}$$

Since $h'(\pi) = 0$ and $h''(\pi) < 0$, $h(\alpha)$ attains a local maximum $h(\pi) = (9 - 8\cos^2\beta)/(4\cos^2\beta)$ at π . It follows from $h(\pi) < h(0)$ that π is not a global maximum point of $h(\alpha)$. Since $h(\alpha)$ is periodic and continuous, its maximum point exists over $(-\pi,\pi)$, thus we may suppose that $h(\alpha)$ attains its maximum at some point α_0 in $(-\pi,\pi)$, then

$$h'(\alpha_0) = 0 \tag{10}$$

and

$$h''(\alpha_0) \le 0. \tag{11}$$

Combining (8) and (10), we may represent $\tan \beta$ in term of α_0 ;

$$\tan \beta = -\frac{5\sin \alpha_0 + 3\sin 2\alpha_0}{3(\cos \alpha_0 + \cos 2\alpha_0)}.$$
 (12)

Substituting it into (9) shows

$$h''(\alpha_0) = -(10\cos\alpha_0 + 12\cos 2\alpha_0) - 2(2\sin 2\alpha_0 + \sin\alpha_0) \frac{5\sin\alpha_0 + 3\sin 2\alpha_0}{\cos\alpha_0 + \cos 2\alpha_0}$$
$$= -\frac{2(11 + 11\cos\alpha_0 + 4\sin^2\alpha_0\cos\alpha_0)}{\cos\alpha_0 + \cos 2\alpha_0}.$$
(13)

Since

$$11 + 11\cos\alpha + 4\sin^2\alpha\cos\alpha > 0$$

whenever $-\pi < \alpha < \pi$, hence from (11) and (13), we deduce that

$$\cos \alpha_0 + \cos 2\alpha_0 > 0$$

which is fulfilled only when $\cos \alpha_0 > 1/2$ i.e. $\alpha_0 \in (-\pi/3, \pi/3)$.

Let $g(\alpha_0)$ denote the quantity given in the right hand side of (12). Since $g'(\alpha) < 0$ over $(-\pi/3, \pi/3)$, there exists one and only one α_0 which satisfies (10) and (11) and $h(\alpha)$ assumes its maximum

$$5 + \frac{9}{4\cos^2\beta} + \frac{13}{1 - 4\sin^2\frac{\alpha_0}{2}}$$

at α_0 .

(8) and (10) also imply

$$\cos\frac{\alpha_0}{2}\left(2\sin\frac{\alpha_0}{2} + 3\frac{\sin\frac{3\alpha_0 + 2\beta}{2}}{\cos\beta}\right) = 0. \tag{14}$$

Since $\alpha_0 \neq \pi$, after letting $x_0 = \sin(\alpha_0/2)$, (14) implies that x_0 is the unique root of the following equation

$$11x - 12x^3 + 3\tan\beta\sqrt{1 - x^2}(1 - 4x^2) = 0.$$

in (-1/2, 1/2). Writing $t_0 = 4x_0^2$ and $t = 4x^2$, we get t_0 is a root of equation (5) in [0, 1).

Let v(t) be the polynomial in the left hand of (5), it is easy to verify that $v(0) \leq 0$, v(1) > 0 and v'(t) > 0 in $0 \leq t < 1$ which together assure the uniqueness of root $t_0 \in [0,1)$ of equation (5).

Therefore Theorem 2 is complete.

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